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## NONLINEAR CONTRACTIONS IN METRIC SPACES UNDER LOCALLY $T$ -TRANSITIVE BINARY RELATIONS

AFTAB ALAM\* AND MOHAMMAD IMDAD\*\*

\*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

E-mail: aafu.amu@gmail.com

\*\*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

E-mail: mhimdad@gmail.com

**Abstract.** In this paper, we present a variant of Boyd-Wong fixed point theorem in a metric space equipped with a locally  $T$ -transitive binary relation, which under universal relation reduces to Boyd-Wong (Proc. Amer. Math. Soc. **20**(1969), 458-464) and Jotic (Indian J. Pure Appl. Math. **26**(1995), 947-952) fixed point theorems. Also, our results extend several other well-known fixed point theorems such as: Alam and Imdad (J. Fixed Point Theory Appl. **17**(2015), no 4, 693-702) and Karapinar and Roldán-López-de-Hierro (J. Inequal. Appl. **2014:522**(2014), 12 pages) besides some others.

**Key Words and Phrases:**  $\mathcal{R}$ -continuity; locally  $T$ -transitive binary relations;  $\varphi$ -contractions;  $\mathcal{R}$ -connected sets.

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### 1. INTRODUCTION

A variety of generalizations of the classical Banach contraction principle [1] is available in the existing literature of metric fixed point theory. These generalizations are obtained in the different directions such as:

- (1) enlarging the class of ambient spaces,
- (2) improving the underlying contraction condition,
- (3) weakening the involved metrical notions (*e.g.* completeness, continuity *etc.*).

Recently, Alam and Imdad [2, 3] obtained an interesting generalization of classical Banach contraction principle by using an amorphous (arbitrary) binary relation. In

doing so, the authors introduced the relation-theoretic analogues of certain involved metrical notions such as: contraction, completeness, continuity *etc.*. In fact, under the universal relation, such newly defined notions reduce to their corresponding usual notions and henceforth relation-theoretic metrical fixed/coincidence point theorems reduce to their corresponding classical fixed/coincidence point theorems (under the universal relation).

Recall that given a nonempty set  $X$ , a subset  $\mathcal{R}$  of  $X^2$  is called a binary relation on  $X$ . For simplicity, we sometimes write  $x\mathcal{R}y$  instead of  $(x, y) \in \mathcal{R}$ . Given  $E \subseteq X$  and a binary relation  $\mathcal{R}$  on  $X$ , the restriction of  $\mathcal{R}$  to  $E$ , denoted by  $\mathcal{R}|_E$ , is defined to be the set  $\mathcal{R} \cap E^2$  (*i.e.*  $\mathcal{R}|_E := \mathcal{R} \cap E^2$ ). Indeed,  $\mathcal{R}|_E$  is a relation on  $E$  induced by  $\mathcal{R}$ .

Out of various classes of binary relations in practice, the following ones are relevant in the present context.

A binary relation  $\mathcal{R}$  on a nonempty set  $X$  is called

- amorphous if  $\mathcal{R}$  has no specific property at all,
- universal if  $\mathcal{R} = X^2$ ,
- empty if  $\mathcal{R} = \emptyset$ ,
- reflexive if  $(x, x) \in \mathcal{R} \forall x \in X$ ,
- symmetric if whenever  $(x, y) \in \mathcal{R}$  then  $(y, x) \in \mathcal{R}$ ,
- antisymmetric if whenever  $(x, y) \in \mathcal{R}$  and  $(y, x) \in \mathcal{R}$  then  $x = y$ ,
- transitive if whenever  $(x, y) \in \mathcal{R}$  and  $(y, z) \in \mathcal{R}$  then  $(x, z) \in \mathcal{R}$ ,
- complete if  $(x, y) \in \mathcal{R}$  or  $(y, x) \in \mathcal{R} \forall x, y \in X$ ,
- partial order if  $\mathcal{R}$  is reflexive, antisymmetric and transitive.

Throughout this paper,  $\mathcal{R}$  stands for a nonempty binary relation but for the sake of simplicity, we write only ‘binary relation’ instead of ‘nonempty binary relation’. Also,  $\mathbb{N}$  stands for the set of natural numbers, while  $\mathbb{N}_0$  for the set of whole numbers (*i.e.*  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ).

The following family of control functions is introduced by Lakshmikantham and Ćirić [4].

$$\Phi = \left\{ \varphi : [0, \infty) \rightarrow [0, \infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \lim_{r \rightarrow t^+} \varphi(r) < t \text{ for each } t > 0 \right\}.$$

With a view to have a self-contained presentation, we recall two fixed point theorems involving nonlinear contractions for class  $\Phi$  using partial order and transitive

binary relations respectively, which have inspired our results in the present paper. In order to understand such results better, we recall firstly the relevant definitions and thereafter state the corresponding results.

**Definition 1.1** [5, 6, 7]. Let  $X$  be a nonempty set equipped with a partial order  $\preceq$ . A self-mapping  $T$  on  $X$  is called increasing or isotone or order-preserving if for any  $x, y \in X$ ,

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

The following notion is formulated by using a suitable property with a view to avoid the necessity of the continuity requirement on the involved mapping specially in the hypotheses of a fixed point theorem due to Nieto and Rodríguez-López [8].

**Definition 1.2** [9]. Let  $(X, d)$  be a metric space equipped with a partial order  $\preceq$ . We say that  $(X, d, \preceq)$  has *ICU* (increasing-convergence-upper bound) property if every increasing convergent sequence in  $X$  is bounded above by its limit (as an upper bound).

The following result, a variant of fixed point theorem of Nieto and Rodríguez-López [8] under  $\varphi$ -contraction, is contained in many papers ( *e.g.* Wu and Liu ([10], Theorem 2.1), Samet *et al.* ([11], Remark 1.3), Kutbi *et al.* ([12], Theorem 5), Karapinar *et al.* ([13], Theorem 10) and Karapinar and Roldán-López-de-Hierro ([14], Theorem 1.2)).

**Theorem 1.3.** *Let  $(X, d)$  be a metric space equipped with a partial order  $\preceq$  and  $T$  a self-mapping on  $X$ . Suppose that the following conditions hold:*

- (a)  $(X, d)$  is complete,
- (b)  $T$  is increasing,
- (c) either  $T$  is continuous or  $(X, d, \preceq)$  has ICU property,
- (d) there exists  $x_0 \in X$  such that  $x_0 \preceq T(x_0)$ ,
- (e) there exists  $\varphi \in \Phi$  such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } x \preceq y.$$

*Then  $T$  has a fixed point. Moreover, if for all  $x, y \in X$ , there exists  $z \in X$  such that  $x \preceq z$  and  $y \preceq z$ , then we obtain uniqueness of the fixed point.*

The above result seems natural but the partial order relation is very restrictive. Samet and Vetro [15] introduced the notion of  $F$ -invariant set and utilized the same to prove some coupled fixed point results for generalized linear contractions in metric spaces without partial order. In 2012, Sintunavarat *et al.* [16] introduced the notion

of transitive property and utilized the same to extend some Samet-Vetro coupled fixed point theorems for nonlinear contractions. On the other hand, Kutbi *et al.* [12] weakened the notion of  $F$ -invariant sets by introducing the notion of  $F$ -closed sets. Recently, Karapinar *et al.* [13] proved some unidimensional versions of earlier coupled fixed point results involving  $F$ -closed sets and then obtained such coupled fixed point results as easy by using their corresponding (unidimensional) fixed point results. As noticed in Alam and Imdad [3], the relation-theoretic metrical fixed/coincidence point theorems combine the idea contained in Karapinar *et al.* [13] as the set  $M$  (utilized by Karapinar *et al.* [13]) being subset of  $X^2$  is in fact a binary relation on  $X$ .

The following notions are unidimensional formulations of transitive property and  $F$ -closed sets.

**Definition 1.4** [13, 17]. We say that a nonempty subset  $M \subseteq X^2$  is

- transitive if  $(x, z) \in M$  for all  $x, y, z \in X$  such that  $(x, y), (y, z) \in M$ .

Given a mapping  $T : X \rightarrow X$ , we say that  $M$  is

- $T$ -transitive if  $(Tx, Ty) \in M$  for all  $x, y, z \in X$  such that  $(Tx, Ty), (Ty, Tz) \in M$ ,
- $T$ -closed if  $(Tx, Ty) \in M$  for all  $x, y \in X$  such that  $(x, y) \in M$ .

**Definition 1.5** [13]. Let  $(X, d)$  be a metric space and let  $M \subseteq X^2$  be a subset. We say that  $(X, d, M)$  is regular if for all sequence  $\{x_n\} \subseteq X$  such that  $x_n \xrightarrow{d} x$  and  $(x_n, x_{n+1}) \in M$  for all  $n \in \mathbb{N}$ , we have  $(x_n, x) \in M$  for all  $n \in \mathbb{N}$ .

The following fixed point theorem indicated in Karapinar and Roldán-López-de-Hierro [14] is a unidimensional version of coupled fixed point theorem of Sintunavarat *et al.* [16].

**Theorem 1.6** [14]. Let  $(X, d)$  be a metric space, let  $T : X \rightarrow X$  be a mapping and let  $M \subseteq X^2$  be a subset such that

- (a)  $(X, d)$  is complete,
- (b)  $M$  is  $T$ -closed and transitive,
- (c) either  $T$  is continuous or  $(X, d, M)$  is regular,
- (d) there exists  $x_0 \in X$  such that  $(x_0, Tx_0) \in M$ ,
- (e) there exists  $\varphi \in \Phi$  such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in M.$$

Then  $T$  has, at least, a fixed point.

The aim of this paper is to extend the Alam-Imdad relation-theoretic fixed point theorem [2] for nonlinear contractions. Our results improve Theorems 1.3 and 1.6 in the following respects:

- (i) the underlying binary relations (partial order or transitive) are replaced by an optimal condition of transitivity namely: locally  $T$ -transitive binary relation,
- (ii) the nonlinear contractive class  $\Phi$  is replaced by relatively enlarger class due to Boyd and Wong [18],
- (iii) the involved metrical notions namely: completeness and continuity are replaced by their  $\mathcal{R}$ -analogues,
- (iv) the  $ICU$  property and the regularity of  $X$  are replaced by relatively weaker notion namely:  $d$ -self-closedness.

## 2. RELATION-THEORETIC NOTIONS AND AUXILIARY RESULTS

In this section, for the sake of completeness, we summarize some necessary definitions and basic results related to our main results.

**Definition 2.1** [2]. Let  $\mathcal{R}$  be a binary relation on a nonempty set  $X$  and  $x, y \in X$ . We say that  $x$  and  $y$  are  $\mathcal{R}$ -comparative if either  $(x, y) \in \mathcal{R}$  or  $(y, x) \in \mathcal{R}$ . We denote it by  $[x, y] \in \mathcal{R}$ .

**Definition 2.2** [19]. Let  $X$  be a nonempty set and  $\mathcal{R}$  a binary relation on  $X$ .

- (1) The inverse or transpose or dual relation of  $\mathcal{R}$ , denoted by  $\mathcal{R}^{-1}$ , is defined by  $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$ .
- (2) The symmetric closure of  $\mathcal{R}$ , denoted by  $\mathcal{R}^s$ , is defined to be the set  $\mathcal{R} \cup \mathcal{R}^{-1}$  (i.e.  $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$ ). Indeed,  $\mathcal{R}^s$  is the smallest symmetric relation on  $X$  containing  $\mathcal{R}$ .

**Proposition 2.3** [2]. For a binary relation  $\mathcal{R}$  defined on a nonempty set  $X$ ,

$$(x, y) \in \mathcal{R}^s \iff [x, y] \in \mathcal{R}.$$

**Definition 2.4** [2]. Let  $X$  be a nonempty set and  $\mathcal{R}$  a binary relation on  $X$ . A sequence  $\{x_n\} \subset X$  is called  $\mathcal{R}$ -preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0.$$

**Definition 2.5** [2]. Let  $X$  be a nonempty set and  $T$  a self-mapping on  $X$ . A binary relation  $\mathcal{R}$  on  $X$  is called  $T$ -closed if for any  $x, y \in X$ ,

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}.$$

**Proposition 2.6** [2]. Let  $X$  be a nonempty set,  $\mathcal{R}$  a binary relation on  $X$  and  $T$  a self-mapping on  $X$ . If  $\mathcal{R}$  is  $T$ -closed, then  $\mathcal{R}^s$  is also  $T$ -closed.

**Proposition 2.7.** Let  $X$  be a nonempty set,  $\mathcal{R}$  a binary relation on  $X$  and  $T$  a self-mapping on  $X$ . If  $\mathcal{R}$  is  $T$ -closed, then, for all  $n \in \mathbb{N}_0$ ,  $\mathcal{R}$  is also  $T^n$ -closed, where  $T^n$  denotes  $n$ th iterate of  $T$ .

**Definition 2.8** [3]. Let  $(X, d)$  be a metric space and  $\mathcal{R}$  a binary relation on  $X$ . We say that  $(X, d)$  is  $\mathcal{R}$ -complete if every  $\mathcal{R}$ -preserving Cauchy sequence in  $X$  converges.

Clearly, every complete metric space is  $\mathcal{R}$ -complete, for any binary relation  $\mathcal{R}$ . Particularly, under the universal relation the notion of  $\mathcal{R}$ -completeness coincides with usual completeness.

**Definition 2.9** [3]. Let  $(X, d)$  be a metric space,  $\mathcal{R}$  a binary relation on  $X$  and  $x \in X$ . A self-mapping  $T$  on  $X$  is called  $\mathcal{R}$ -continuous at  $x$  if for any  $\mathcal{R}$ -preserving sequence  $\{x_n\}$  such that  $x_n \xrightarrow{d} x$ , we have  $T(x_n) \xrightarrow{d} T(x)$ . Moreover,  $T$  is called  $\mathcal{R}$ -continuous if it is  $\mathcal{R}$ -continuous at each point of  $X$ .

Clearly, every continuous mapping is  $\mathcal{R}$ -continuous, for any binary relation  $\mathcal{R}$ . Particularly, under the universal relation the notion of  $\mathcal{R}$ -continuity coincides with usual continuity.

The following notion is a generalization of  $d$ -self-closedness of a partial order relation  $(\preceq)$  (defined by Turinici [20, 21]).

**Definition 2.10** [2]. Let  $(X, d)$  be a metric space. A binary relation  $\mathcal{R}$  on  $X$  is called  $d$ -self-closed if for any  $\mathcal{R}$ -preserving sequence  $\{x_n\}$  such that  $x_n \xrightarrow{d} x$ , there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  with  $[x_{n_k}, x] \in \mathcal{R} \forall k \in \mathbb{N}_0$ .

**Definition 2.11** [22]. Let  $X$  be a nonempty set and  $\mathcal{R}$  a binary relation on  $X$ . A subset  $E$  of  $X$  is called  $\mathcal{R}$ -directed if for each pair  $x, y \in E$ , there exists  $z \in X$  such that  $(x, z) \in \mathcal{R}$  and  $(y, z) \in \mathcal{R}$ .

**Definition 2.12** [23]. Let  $X$  be a nonempty set and  $\mathcal{R}$  a binary relation on  $X$ . For  $x, y \in X$ , a path of length  $k$  (where  $k$  is a natural number) in  $\mathcal{R}$  from  $x$  to  $y$  is a finite sequence  $\{z_0, z_1, z_2, \dots, z_k\} \subset X$  satisfying the following conditions:

- (i)  $z_0 = x$  and  $z_k = y$ ,
- (ii)  $(z_i, z_{i+1}) \in \mathcal{R}$  for each  $i$  ( $0 \leq i \leq k-1$ ).

Notice that a path of length  $k$  involves  $k+1$  elements of  $X$ , although they are not necessarily distinct.

**Definition 2.13** [3]. Let  $X$  be a nonempty set and  $\mathcal{R}$  a binary relation on  $X$ . A subset  $E$  of  $X$  is called  $\mathcal{R}$ -connected if for each pair  $x, y \in E$ , there exists a path in  $\mathcal{R}$  from  $x$  to  $y$ .

Given a binary relation  $\mathcal{R}$  and a self-mapping  $T$  on a nonempty set  $X$ , we use the following notations.

- (i)  $F(T)$  := the set of all fixed points of  $T$ ,
- (ii)  $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$ .

The following result is the relation-theoretic version of Banach contraction principle.

**Theorem 2.14** [2, 3]. Let  $(X, d)$  be a metric space,  $\mathcal{R}$  a binary relation on  $X$  and  $T$  a self-mapping on  $X$ . Suppose that the following conditions hold:

- (a)  $(X, d)$  is  $\mathcal{R}$ -complete,
- (b)  $\mathcal{R}$  is  $T$ -closed,
- (c) either  $T$  is  $\mathcal{R}$ -continuous or  $\mathcal{R}$  is  $d$ -self-closed,
- (d)  $X(T, \mathcal{R})$  is nonempty,
- (e) there exists  $\alpha \in [0, 1)$  such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then  $T$  has a fixed point. Moreover if  $X$  is  $\mathcal{R}^s$ -connected then  $T$  has a unique fixed point.

Now, we re-define the notion of  $T$ -transitivity employed in Definition 1.4 in the framework of binary relation.

**Definition 2.15.** Let  $X$  be a nonempty set and  $T$  a self-mapping on  $X$ . A binary relation  $\mathcal{R}$  on  $X$  is called  $T$ -transitive if for any  $x, y, z \in X$ ,

$$(Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}.$$

Inspired by Turinici [24, 25], we introduce the following notion by localizing the notion of transitivity.

**Definition 2.16.** A binary relation  $\mathcal{R}$  on a nonempty set  $X$  is called locally transitive if for each (effectively)  $\mathcal{R}$ -preserving sequence  $\{x_n\} \subset X$  (with range  $E := \{x_n : n \in \mathbb{N}_0\}$ ), the binary relation  $\mathcal{R}|_E$  is transitive.

Henceforth, the notions of  $T$ -transitivity and locally transitivity both are relatively weaker than the notion of transitivity but they are independent of each others. In order to make them compatible, we introduce the following notion of transitivity.

**Definition 2.17.** Let  $X$  be a nonempty set and  $T$  a self-mapping on  $X$ . A binary relation  $\mathcal{R}$  on  $X$  is called locally  $T$ -transitive if for each (effectively)  $\mathcal{R}$ -preserving sequence  $\{x_n\} \subset T(X)$  (with range  $E := \{x_n : n \in \mathbb{N}_0\}$ ), the binary relation  $\mathcal{R}|_E$  is transitive.

The following result establishes the superiority of locally  $T$ -transitivity over other types of transitivity.

**Proposition 2.18.** Let  $X$  be a nonempty set,  $\mathcal{R}$  a binary relation on  $X$  and  $T$  a self-mapping on  $X$ .

- (i)  $\mathcal{R}$  is  $T$ -transitive  $\Leftrightarrow \mathcal{R}|_{T(X)}$  is transitive.
- (ii)  $\mathcal{R}$  is locally  $T$ -transitive  $\Leftrightarrow \mathcal{R}|_{T(X)}$  is locally transitive.
- (iii)  $\mathcal{R}$  is transitive  $\Rightarrow \mathcal{R}$  is locally transitive  $\Rightarrow \mathcal{R}$  is locally  $T$ -transitive.
- (iv)  $\mathcal{R}$  is transitive  $\Rightarrow \mathcal{R}$  is  $T$ -transitive  $\Rightarrow \mathcal{R}$  is locally  $T$ -transitive.

The following family of control functions is indicated in Boyd and Wong [18] but was later used in Jotic [26].

$$\Omega = \left\{ \varphi : [0, \infty) \rightarrow [0, \infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t^+} \varphi(r) < t \text{ for each } t > 0 \right\}.$$

It is clear that the class  $\Omega$  enlarges the class  $\Phi$ , i.e.,  $\Phi \subset \Omega$ .



**Proposition 2.19.** *If  $(X, d)$  is a metric space,  $\mathcal{R}$  is a binary relation on  $X$ ,  $T$  is a self-mapping on  $X$  and  $\varphi \in \Omega$ , then the following contractivity conditions are equivalent:*

- (I)  $d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R},$
- (II)  $d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } [x, y] \in \mathcal{R}.$

We skip the proof of above proposition as it is similar to that of Proposition 2.3 [2].

Finally, we record the following known results, which are needed in the proof of our main results.

**Lemma 2.20** [9]. *Let  $\varphi \in \Omega$ . If  $\{a_n\} \subset (0, \infty)$  is a sequence such that  $a_{n+1} \leq \varphi(a_n) \quad \forall n \in \mathbb{N}_0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

**Lemma 2.21** [27, 25]. *Let  $(X, d)$  be a metric space and  $\{x_n\}$  a sequence in  $X$ . If  $\{x_n\}$  is not a Cauchy, then there exist  $\epsilon > 0$  and two subsequences  $\{x_{n_k}\}$  and  $\{x_{m_k}\}$  of  $\{x_n\}$  such that*

- (i)  $k \leq m_k < n_k \quad \forall k \in \mathbb{N},$
- (ii)  $d(x_{m_k}, x_{n_k}) > \epsilon \quad \forall k \in \mathbb{N},$
- (iii)  $d(x_{m_k}, x_{n_{k-1}}) \leq \epsilon \quad \forall k \in \mathbb{N}.$

Moreover, suppose that  $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$ , then

- (iv)  $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \epsilon,$
- (v)  $\lim_{k \rightarrow \infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon.$

### 3. FIXED POINT THEOREMS

Firstly, we prove a result on the existence of fixed points under  $\varphi$ -contractivity condition, which runs as follows.

**Theorem 3.1.** *Let  $(X, d)$  be a metric space,  $\mathcal{R}$  a binary relation on  $X$  and  $T$  a self-mapping on  $X$ . Suppose that the following conditions hold:*

- (a)  $(X, d)$  is  $\mathcal{R}$ -complete,
- (b)  $\mathcal{R}$  is  $T$ -closed and locally  $T$ -transitive,
- (c) either  $T$  is  $\mathcal{R}$ -continuous or  $\mathcal{R}$  is  $d$ -self-closed,
- (d)  $X(T, \mathcal{R})$  is nonempty,
- (e) there exists  $\varphi \in \Omega$  such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then  $T$  has a fixed point.

**Proof.** In view of assumption (d), take arbitrarily  $x_0 \in X(T, \mathcal{R})$ . Construct the sequence  $\{x_n\}$  of Picard iterates based at the initial point  $x_0$ , *i.e.*,

$$x_n = T^n(x_0) \quad \forall n \in \mathbb{N}_0. \quad (1)$$

As  $(x_0, Tx_0) \in \mathcal{R}$ , using  $T$ -closedness of  $\mathcal{R}$  and Proposition 2.7, we obtain

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$$

so that

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (2)$$

Thus the sequence  $\{x_n\}$  is  $\mathcal{R}$ -preserving. Applying the contractivity condition (e) to (2), we deduce, for all  $n \in \mathbb{N}_0$  that

$$d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1})).$$

Hence by Lemma 2.20, we obtain

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (3)$$

Now, we show that  $\{x_n\}$  is a Cauchy sequence. On contrary, suppose that  $\{x_n\}$  is not Cauchy. Therefore, owing to Lemma 2.21, there exist  $\epsilon > 0$  and two subsequences  $\{x_{n_k}\}$  and  $\{x_{m_k}\}$  of  $\{x_n\}$  such that

$$k \leq m_k < n_k, \quad d(x_{m_k}, x_{n_k}) > \epsilon \geq d(x_{m_k}, x_{n_k-1}) \quad \forall k \in \mathbb{N}. \quad (4)$$

Further, in view of (3), Lemma 2.21 assures us that

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \lim_{k \rightarrow \infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon. \quad (5)$$

Denote  $r_k := d(x_{m_k}, x_{n_k})$ . As  $\{x_n\}$  is  $\mathcal{R}$ -preserving (owing to (2)) and  $\{x_n\} \subset T(X)$  (owing to (1)), by locally  $T$ -transitivity of  $\mathcal{R}$ , we have  $(x_{m_k}, x_{n_k}) \in \mathcal{R}$ . Hence, applying contractivity condition (e), we obtain

$$\begin{aligned} d(x_{m_k+1}, x_{n_k+1}) &= d(Tx_{m_k}, Tx_{n_k}) \\ &\leq \varphi(d(x_{m_k}, x_{n_k})). \\ &= \varphi(r_k) \end{aligned}$$

so that

$$d(x_{m_k+1}, x_{n_k+1}) \leq \varphi(r_k). \quad (6)$$

Using the facts that  $r_k \rightarrow \epsilon$  in the real line as  $k \rightarrow \infty$  (owing to (5)) and  $r_k > \epsilon \forall k \in \mathbb{N}$  (owing to (4)) and by the definition of  $\Omega$ , we have

$$\limsup_{k \rightarrow \infty} \varphi(r_k) = \limsup_{r \rightarrow \epsilon^+} \varphi(r) < \epsilon. \quad (7)$$

On taking limit superior as  $k \rightarrow \infty$  in (6) and using (5) and (7), we obtain

$$\epsilon = \limsup_{k \rightarrow \infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup_{k \rightarrow \infty} \varphi(r_k) < \epsilon,$$

which is a contradiction. Therefore,  $\{x_n\}$  is a Cauchy sequence. Hence,  $\{x_n\}$  is an  $\mathcal{R}$ -preserving Cauchy sequence. By  $\mathcal{R}$ -completeness of  $X$ ,  $\exists x \in X$  such that  $x_n \xrightarrow{d} x$ .

Finally, we use assumption (c) to show that  $x$  is a fixed point of  $T$ . Suppose that  $T$  is  $\mathcal{R}$ -continuous. As  $\{x_n\}$  is  $\mathcal{R}$ -preserving with  $x_n \xrightarrow{d} x$ ,  $\mathcal{R}$ -continuity of  $T$  implies that  $x_{n+1} = T(x_n) \xrightarrow{d} T(x)$ . Using the uniqueness of limit, we obtain  $T(x) = x$ , *i.e.*,  $x$  is a fixed point of  $T$ .

Alternately, assume that  $\mathcal{R}$  is  $d$ -self-closed. Again as  $\{x_n\}$  is  $\mathcal{R}$ -preserving such that  $x_n \xrightarrow{d} x$ ,  $d$ -self-closedness of  $\mathcal{R}$  guarantees the existence of a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  with  $[x_{n_k}, x] \in \mathcal{R} \forall k \in \mathbb{N}_0$ . On using the fact  $[x_{n_k}, x] \in \mathcal{R}$ , assumption (e) and Proposition 2.19, we obtain

$$d(x_{n_k+1}, Tx) = d(Tx_{n_k}, Tx) \leq \varphi(d(x_{n_k}, x)) \forall k \in \mathbb{N}_0.$$

We claim that

$$d(x_{n_k+1}, Tx) \leq d(x_{n_k}, x) \forall k \in \mathbb{N}. \quad (8)$$

On account of two different possibilities arising here, we consider a partition  $\{\mathbb{N}^0, \mathbb{N}^+\}$  of  $\mathbb{N}$ , *i.e.*,  $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$  and  $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$  verifying that

- (i)  $d(x_{n_k}, x) = 0 \forall k \in \mathbb{N}^0$ ,
- (ii)  $d(x_{n_k}, x) > 0 \forall k \in \mathbb{N}^+$ .

In case (i), we have  $d(Tx_{n_k}, Tx) = 0 \forall k \in \mathbb{N}^0$ , which implies that  $d(x_{n_k+1}, Tx) = 0 \forall k \in \mathbb{N}^0$  and hence (8) holds for all  $k \in \mathbb{N}^0$ . In case (ii), by the definition of  $\Omega$ , we have  $d(x_{n_k+1}, Tx) \leq \varphi(d(x_{n_k}, x)) < d(x_{n_k}, x) \forall k \in \mathbb{N}^+$  and hence (8) holds for all  $k \in \mathbb{N}^+$ . Thus (8) holds for all  $k \in \mathbb{N}$ .

Taking limit of (8) as  $k \rightarrow \infty$  and using  $x_{n_k} \xrightarrow{d} x$ , we obtain  $x_{n_k+1} \xrightarrow{d} T(x)$ . Owing to the uniqueness of limit, we obtain  $T(x) = x$  so that  $x$  is a fixed point of  $T$ .

Using Proposition 2.18, we obtain the following consequence of Theorem 3.1.

**Corollary 3.2.** *Theorem 3.1 remains true if locally  $T$ -transitivity of  $\mathcal{R}$  (utilized in assumption (b)) is replaced by any one of the following conditions (besides retaining rest of the hypotheses):*

- (i)  $\mathcal{R}$  is transitive,
- (ii)  $\mathcal{R}$  is  $T$ -transitive,
- (iii)  $\mathcal{R}$  is locally transitive.

Now, we prove a corresponding uniqueness result.

**Theorem 3.3.** *In addition to the hypotheses of Theorem 3.1, suppose that the following condition holds:*

- (u)  $T(X)$  is  $\mathcal{R}^s$ -connected.

*Then  $T$  has a unique fixed point.*

**Proof.** In view of Theorem 3.1,  $F(T) \neq \emptyset$ . Take  $x, y \in F(T)$ , then for all  $n \in \mathbb{N}_0$ , we have

$$T^n(x) = x \text{ and } T^n(y) = y. \quad (9)$$

Clearly  $x, y \in T(X)$ . By assumption (u), there exists a path (say  $\{z_0, z_1, z_2, \dots, z_k\}$ ) of some finite length  $k$  in  $\mathcal{R}^s$  from  $x$  to  $y$  so that

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (10)$$

As  $\mathcal{R}$  is  $T$ -closed, using Propositions 2.6 and 2.7, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \quad (11)$$

Now, for each  $n \in \mathbb{N}_0$  and for each  $i \ (0 \leq i \leq k-1)$ , define  $t_n^i := d(T^n z_i, T^n z_{i+1})$ . We claim that

$$\lim_{n \rightarrow \infty} t_n^i = 0. \quad (12)$$

Fix  $i$  and distinguish two cases. Firstly, suppose that  $t_{n_0}^i = d(T^{n_0} z_i, T^{n_0} z_{i+1}) = 0$  for some  $n_0 \in \mathbb{N}_0$ , i.e.  $T^{n_0}(z_i) = T^{n_0}(z_{i+1})$ , which implies that  $T^{n_0+1}(z_i) = T^{n_0+1}(z_{i+1})$ . Consequently, we get  $t_{n_0+1}^i = d(T^{n_0+1} z_i, T^{n_0+1} z_{i+1}) = 0$ . Thus by induction, we get  $t_n^i = 0 \ \forall n \geq n_0$ , yielding thereby  $\lim_{n \rightarrow \infty} t_n^i = 0$ . On the other hand, suppose that  $t_n > 0 \ \forall n \in \mathbb{N}_0$ , then on using (11), assumption (e) and Proposition 2.19, we obtain

$$\begin{aligned} t_{n+1}^i &= d(T^{n+1} z_i, T^{n+1} z_{i+1}) \\ &\leq \varphi(d(T^n z_i, T^n z_{i+1})) \\ &= \varphi(t_n^i) \end{aligned}$$

so that

$$t_{n+1}^i \leq \varphi(t_n^i).$$

Hence, on applying Lemma 2.20, we obtain  $\lim_{n \rightarrow \infty} t_n^i = 0$ . Thus, in both the cases, (12) is proved for each  $i$  ( $0 \leq i \leq k-1$ ).

Making use of (9), (10), (12) and the triangular inequality, we obtain

$$d(x, y) = d(T^n z_0, T^n z_k) \leq t_n^0 + t_n^1 + \cdots + t_n^{k-1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that  $x = y$ . Hence  $T$  has a unique fixed point.

The following consequence of Theorem 3.3 is worth recording.

**Corollary 3.4.** *Theorem 3.3 remains true if we replace the condition (u) by one of the following conditions (besides retaining rest of the hypotheses):*

(u')  $\mathcal{R}|_{T(X)}$  is complete,

(u'')  $T(X)$  is  $\mathcal{R}^s$ -directed.

**Proof.** If (u') holds, then for each  $u, v \in T(X)$ ,  $[u, v] \in \mathcal{R}$ , which amounts to say that  $\{u, v\}$  is a path of length 1 in  $\mathcal{R}^s$  from  $u$  to  $v$ . Hence  $T(X)$  is  $\mathcal{R}^s$ -connected consequently Theorem 3.3 gives rise the conclusion.

Otherwise, if (u'') holds then for each  $u, v \in T(X)$ ,  $\exists z \in X$  such that  $[u, z] \in \mathcal{R}$  and  $[v, z] \in \mathcal{R}$ , which amounts to say that  $\{u, z, v\}$  is a path of length 2 in  $\mathcal{R}^s$  from  $u$  to  $v$ . Hence  $T(X)$  is  $\mathcal{R}^s$ -connected and again by Theorem 3.3 the conclusion is immediate.

Now, we consider some special cases, wherein our results deduce several well-known fixed point theorems of the existing literature.

- (1) Under the universal relation (*i.e.*  $\mathcal{R} = X^2$ ), Theorem 3.3 deduces to the Jotic fixed point theorem proved in [26], which is a generalization of Boyd-Wong fixed point theorem [18].
- (2) On setting  $\mathcal{R} = \preceq$ , the partial order in Theorem 3.1 as well as Corollary 3.4, we obtain Theorem 1.3. Clearly,  $T$ -closedness of  $\preceq$  is equivalent to increasing property of  $T$ .
- (3) Taking  $\mathcal{R} = M$ , the transitive binary relation in Theorem 3.1, we obtain Theorem 1.6.

**Conclusion:** In order to ensure the existence of fixed points for linear contraction mapping  $T$ , the underlying binary relation is required to be  $T$ -closed (see Theorem 2.14). But whenever, we extend Theorem 2.14 from linear contractions to Boyd-Wong type nonlinear contractions then this restriction on the underlying binary relation is not enough. We additionally do require locally  $T$ -transitivity of  $\mathcal{R}$ , which

substantiates the utility of this extension. As possible problems, authors encourage the researchers of this domain to prove such results for other types of contractions.

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